Logarithm properties

Note Title 14/09/2008

See Sec 1.6 in Stewart.

$\ln(ab) = \ln a + \ln b \qquad a>0, b>0$

$\ln(a^b) = b \ln a \qquad a>0$

$\ln e = 1$

$\ln 1 = 0$

$\ln 0^+ = -\infty \qquad \left(\lim_{x\to 0^+} \ln x = -\infty . \right)$

Other bases: $\lg x = \log_2 x$

$\log_{10} x$ "common"

Change of basis

$y = \log_a x \iff a^y = x$

but $a = b^{\log_b a}$

so $a^y = (b^{\log_b a})^y = b^{y \log_b a}$

So $x = b^{y \log_b a}$ hence $\log_b x = y \log_b a$

or
$$y = \log_a x = \frac{\log_b x}{\log_b a}$$

Example: $\ln 2 = 0.69314718056\ldots$

$$\log_{10} 2 = \frac{\ln 2}{\ln 10}$$

$$\ln 10 = 2.302585093\ldots$$

$$\therefore \log_{10} 2 = \frac{0.69314718056\ldots}{2.302585093\ldots}$$

$$= 0.3010299956\ldots$$

Check: $10^{0.30103} \doteq 2.000000002$

success.

Most often used to convert to ln.

$$\ln x = \frac{\log_b x}{\log_b e}.$$ (uncommon anyway)

A common BAD error

$$\ln(a+b) \neq \ln a + \ln b$$

Check: $\ln(1+0) = 0$

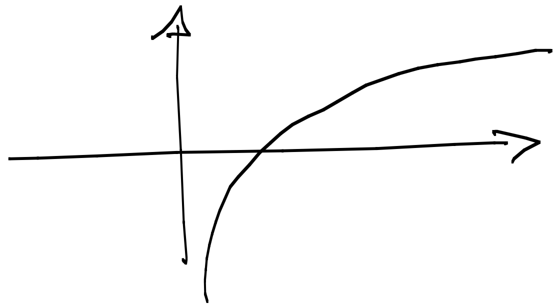
$$\neq \ln(1) + \ln 0$$
$$= 0 + (-\infty)$$

$$\ln(1+1) = \ln 2 = 0.693\ldots$$
$$\neq \ln 1 + \ln 1 = 0 + 0 = 0.$$

Excercise: if $f(x)$ is such that $f(a+b) = f(a) + f(b)$, what <u>could</u> f be?

Growth properties

① $\ln x$ increases as x increases



higher to the right.

(this is because e^x increases as x increases).

② $\ln x$ grows "tediously slowly" (in mem. J.B. Ehrman)

x	$\ln x$	$x^{1/200}$
1	0	1
1,000	6.9077	1.035
1,000,000	13.815	1.071
1,000,000,000	20.723	1.109
10^{12}	27.631	1.148
10^{100}	230.258	3.162
10^{10000}	23,025.8	10^{50}

(the inverse property: e^x grows more rapidly than any x^n, eventually).

http://en.wikipedia.org/wiki/Exponential_growth

These facts will be proved in general later, using limits.

Plotting on a log scale:

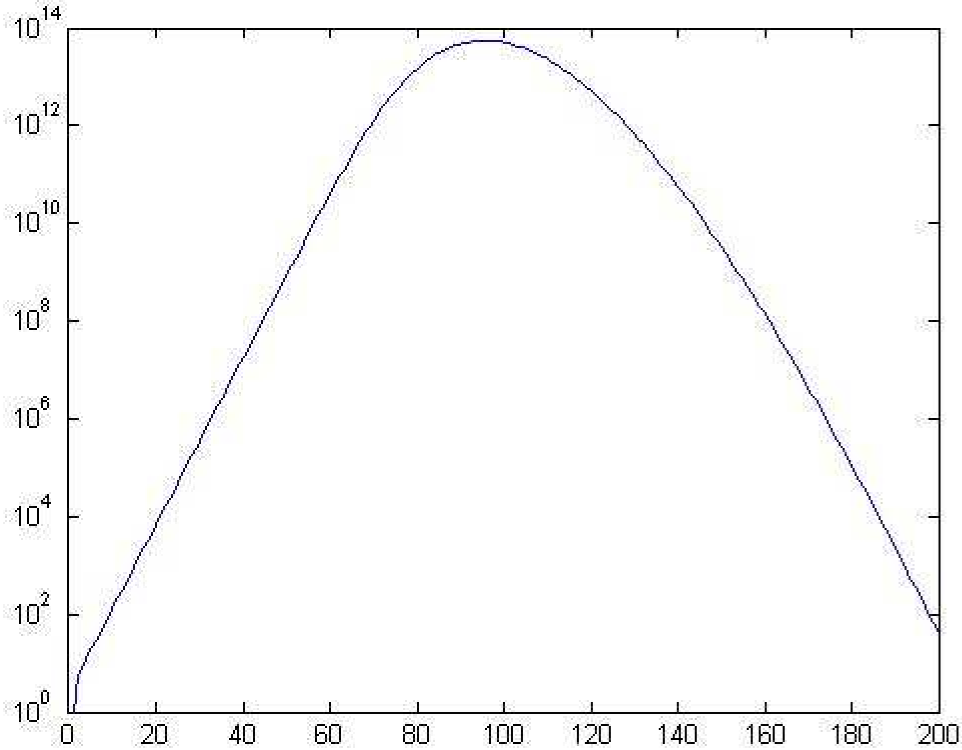
If we plot $\log_{10} f(x)$ vs x but place tickmarks at (say)

$-2 = \log_{10}(0.01)$, $-1 = \log_{10} 0.1$,

$0 = \log_{10} 1$, $1 = \log_{10} 10$, $2 = \log_{10} 100$

and so on, we get a semilog (y) plot.

A plot of something that $\to 0$ eventually

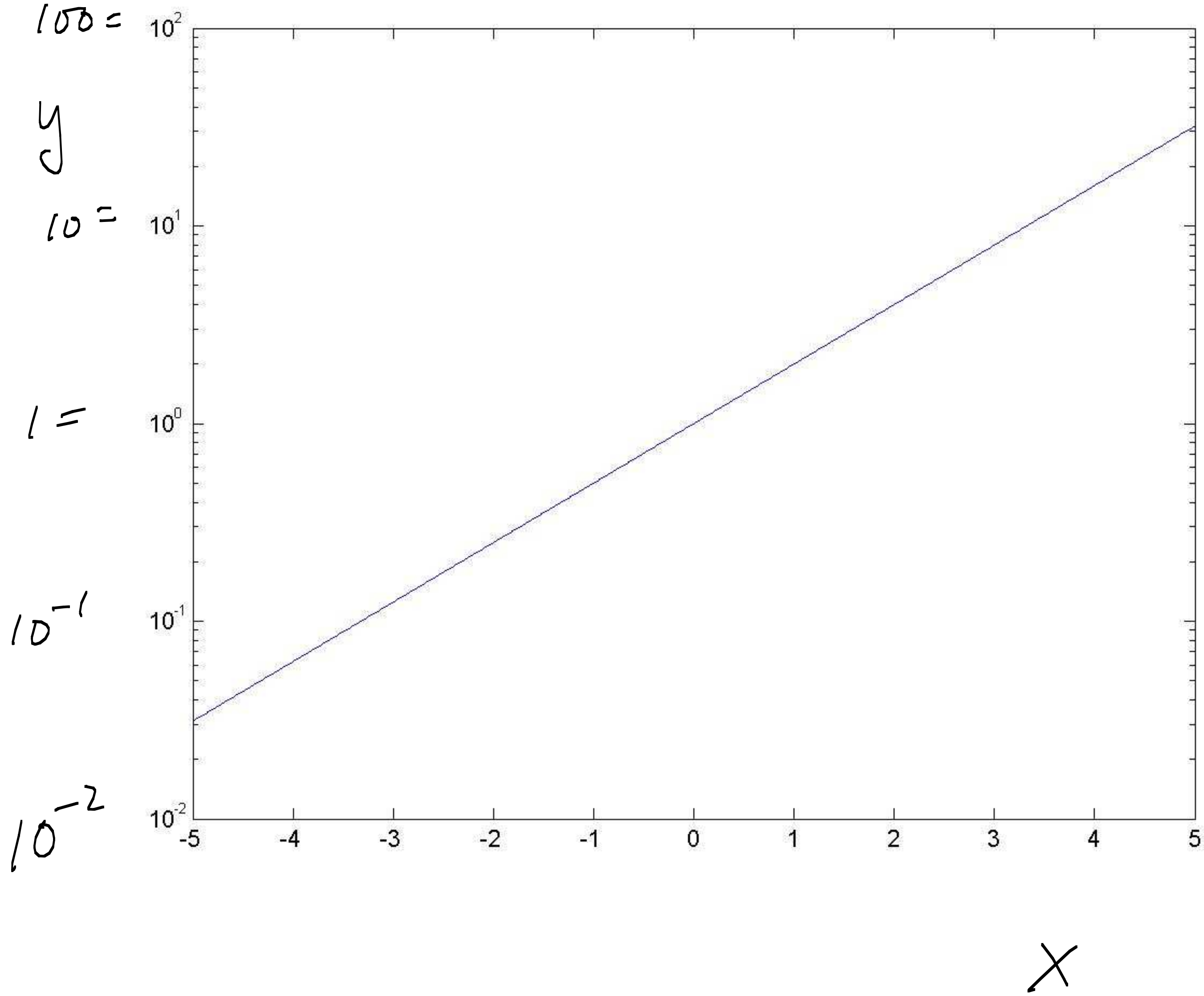


(produced by matlab)

We label the tickmarks with the un-logarithmed values, & then we can compress widely-varying data onto the same plot.

Plotting 2^x on such a plot gives a straight line.

$$y = 2^x$$



A trick with $\ln x$.

$$e^x \doteq 1 + x \quad \text{if } x \text{ is small.}$$

$$x^{1/2048} = e^{\ln x / 2048} \doteq 1 + \frac{\ln x}{2048}$$

if x isn't too big.

note $2048 = 2^{11}$

and we can compute $x^{1/2048}$

by taking 11 square roots in

succession:

$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{x}}}}}}}}}}} = x^{1/2048}$$

$$= \left(\left(\left(\left(\left(\left(\left(\left(\left(\left(x^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}\right)^{1/2}$$

So, to find (say) $\ln 17$ on a

calculator that has $\sqrt{\ }$ but not $\ln$:

(eg Windows accessory)

(1) • $17^{1/2048} \doteq 1.001384$

(2) • Subtract 1, what's left is $\approx \frac{\ln 17}{2048}$

(3) • multiply by 2048

$\ln 17 \doteq 2.8351$

(true answer: $\ln 17 \doteq 2.8332$, not bad).